Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Some Properties of a Generalized Hypergeometric Function.

By F. H. Jackson.

In Heine's Kugelfunctionen, Vol. I, Appendix to Chapter 2, some properties of the series

$$1+\frac{(1-a)(1-b)}{(1-q)(1-c)}x+\frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}x^2 + \ldots . \equiv \phi\,[a,\, b,\, c,\, q,\, x]$$

are given. The three-fold symmetry of this series is investigated by Professor L. J. Rogers, p. 171, Vol. XXIV, Proc. London Math. Soc. In this paper some properties of the more general series

$$1+\frac{[\alpha][\beta]}{[l]\,[\gamma]}\lambda x^{[l]}+\frac{[\alpha][\alpha+l][\beta][\beta+l]}{[l][2l][\gamma][\gamma+l]}\lambda^2 x^{[2l]} + \ldots . \equiv F([\alpha][\beta][\gamma]\lambda x) \quad (1)$$

will be investigated.

Here $[\alpha]$ denotes $\frac{p^{\alpha}-1}{p-1}$.

Owing to the form of the indices of the element x, namely, $[l]\,.\,[2l]$, etc., a differential equation for F can be found analogous to the differential equation of the Hypergeometric Series :

F reduces to $\phi\,[a,\, b,\, c,\, q,\, x]$ if for p^{α} we substitute a;

p^{β} " " b;

p^{γ} " " c;

p^{l} " " q;

λ " " x;

x " " 1.

The differential equation satisfied by F is

$$\begin{aligned}\lambda[\alpha][\beta]\,.\,F+\{\lambda p^{\alpha}[\beta+l]\,x^{[l]}+\lambda p^{\beta}[\alpha]\,x^{[l]}-[\gamma]\}\,\frac{dF}{d(x)}&\\ +\{\lambda p^{\alpha+\beta+l}x^{[2l]}-p^{\gamma}x^{p^{l}[l]}\}\,\frac{d^2F}{d^2(x)}&\\ =\lambda[\alpha][\beta]\,\{F([\alpha+l][\beta+l][\gamma]\,.\,\lambda\,.\,x)&\\ -F([\alpha+l][\beta+l][\gamma]\,\lambda\,.\,x^{p^{l}})\},& \qquad (2)\end{aligned}$$

in which

$$\frac{d}{d(x)} \text{ is } \frac{d}{d(x^{[l]})},$$

$$\frac{d^2}{d^2(x)} \text{ is } \frac{d}{d(x^{p^{l}})}\cdot\frac{d}{d(x)},$$

differentiation being with regard to $x^{[l]}$, $x^{p^{l}[l]}$, $x^{p^{2l}[l]}$, etc., successively as independent variables.

That F satisfies the above equation can be shown by a method analogous to the ordinary method for integrating equations in series. See Proc. London Math. Soc., series 2, Vol. I, page 83.

The principal results to be obtained in the following analysis are

$$F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta})=\frac{\Gamma([\gamma-\alpha-\beta])\,\Gamma([\gamma])}{\Gamma([\gamma-\alpha])\,\Gamma([\gamma-\beta])}, \qquad (3)$$

$$F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l})=\frac{\Gamma([\gamma-\alpha-\beta])\,\Gamma([\gamma])}{\Gamma([\gamma-\alpha])\,\Gamma([\gamma-\beta])}\cdot\frac{\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}}{[\gamma-\alpha-\beta-l]}; \qquad (4)$$

Γ does not denote Euler's Gamma-Function but a certain convergent infinite product which reduces when $p=1$ to Euler's Gamma-Function. Theorem (3) is well known and is due to Heine. The second theorem is, I think, new.

Differentiating the series $F([\alpha][\beta][\gamma]\,.\,\lambda x)$, term by term, we obtain

$$\frac{dF}{d(x)}=\frac{[\alpha][\beta]}{[\gamma]}\,\lambda\,F([\alpha+l][\beta+l][\gamma+l]\,\lambda x^{p^{l}})\,.$$

Also $F([\alpha][\beta][\gamma]\,\lambda_1 x)-F([\alpha][\beta][\gamma-l]\,\lambda x)$ is a series in which the term involving $x^{[ml]}$ is

$$\frac{[\alpha][\alpha+l]\,\ldots\ldots\,[\alpha+(m-1)\,l]\,.\,[\beta][\beta+l]\,\ldots\ldots\,[\beta+(m-1)\,l]}{[l][2l]\,\ldots\ldots\,[ml]\,.\,[\gamma][\gamma+l]\,\ldots\ldots\,[\gamma+(m-1)\,l]}$$

$$\times\,\lambda_1^{m}\left\{1-\frac{\lambda^{m}}{\lambda_1^{m}}\cdot\frac{[\gamma+(m-1)\,l]}{[\gamma-l]}\right\}\,x^{[ml]}\,.$$

If $\lambda_1 = \lambda$ this reduces to

$$-\frac{[\alpha][\alpha+l]\ldots\ldots[\alpha+(m-1)l]\,.\,[\beta][\beta+l]\ldots\ldots[\beta+(m-1)l]}{[l][2l]\ldots\ldots[(m-1)l]\,.\,[\gamma-l][\gamma]\ldots\ldots[\gamma+(m-1)l]}\,p^{\gamma-l}\lambda^m x^{[ml]},$$

which is the general term of

$$-\frac{p^{\gamma-l}[\alpha][\beta]}{[\gamma][\gamma-l]}\lambda x^{[l]}\,.\,F([\alpha+l][\beta+l][\gamma+l]\,\lambda x^p),$$

that is of

$$-\frac{p^{\gamma-l}x^{[l]}}{[\gamma-l]}\,\frac{dF}{d(x)}.$$

So we have

$$F([\alpha][\beta][\gamma]\,\lambda x) - F([\alpha][\beta][\gamma-l]\,\lambda x) = -\frac{p^{\gamma-l}x^{[l]}}{[\gamma-l]}\cdot\frac{dF}{d(x)}. \qquad (5)$$

Similarly we may show that

$$F([\alpha][\beta][\gamma],\,\lambda p^l,\,x) - F([\alpha][\beta][\gamma-l]\,\lambda x) = -\frac{x^{[l]}}{[\gamma-l]}\cdot\frac{dF}{d(x)}. \qquad (6)$$

Now in the differential equation

$$\begin{aligned}\lambda[\alpha][\beta]F + \{\lambda p^{\alpha}[\beta+l]x^{[l]} + \lambda p^{\beta}[\alpha]x^{[l]} - [\gamma]\}\,\frac{dF}{d(x)} \\ + \{\lambda p^{\alpha+\beta+l}x^{[2l]} - p^{\lambda}x^{p^l[l]}\}\,\frac{d^2F}{d^2(x)} \\ = [\alpha][\beta]\lambda\{F([\alpha+l][\beta+l][\gamma]\,\lambda x) - F([\alpha+l][\beta+l][\gamma]\,\lambda x^{p^l})\}, \qquad (7)\end{aligned}$$

putting $x=1$ the right side vanishes identically and $\left[\frac{d^2F}{d^2(x)}\right]_{x=1}$ is finite, provided $p^l < 1$;
so choosing $\lambda = p^{\gamma-\alpha-\beta-l}$ to make the bracket coefficient of $\frac{d^2F}{d^2(x)}$ vanish we obtain from the differential equation

$$\begin{aligned}p^{\gamma-\alpha-\beta-l}[\alpha][\beta]F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l}) \\ + \{p^{\gamma-\beta-l}[\beta+l] + p^{\gamma-\alpha-l}[\alpha] - [\gamma]\}\left[\frac{dF}{d(x)}\right]_{x=1} = 0; \qquad (8)\end{aligned}$$

putting $\lambda = p^{\gamma-\alpha-\beta-l}$ in (5) and eliminating $\left[\frac{dF}{d(x)}\right]_{x=1}$ between (5) and (8) we have a relation

$$F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l})$$
$$=\frac{[\gamma-l]\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}}{[\gamma-\alpha-l][\gamma-\beta-l]}\,F([\alpha][\beta][\gamma-l]\,p^{\gamma-\alpha-\beta-l}), \qquad (9)$$

a relation between quasi-successive functions. This relation does not enable us to express $F([\alpha][\beta][\gamma]\ p^{\gamma-\alpha-\beta-l}$ in the form of an infinite product as would have been the case if the λ (namely $p^{\gamma-\alpha-\beta-l}$) in the F on the right side had been $p^{\gamma-\alpha-\beta-2l}$. When p is made unity $\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}$ reduces to $(\gamma-\alpha-\beta-l)$ and we have

$$F(\alpha\beta\gamma)=\frac{(\gamma-1)(\gamma-\alpha-\beta-1)}{(\gamma-\alpha-\beta)(\gamma-\beta-1)}\,F(\alpha\,.\,\beta\,.\,\gamma-1),$$

from which $F(\alpha\beta\gamma)$ may be expressed as an infinite product. To obtain a relation between two successive functions $F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta})$ and $F([\alpha][\beta][\gamma-l]\,p^{\gamma-\alpha-\beta-l})$ we eliminate $\dfrac{dF}{d(x)}$ between

$$F([\alpha][\beta][\gamma]\,\lambda x)\qquad-F([\alpha][\beta][\gamma-l]\,\lambda x)=-\frac{p^{\gamma-l}x^{[l]}}{[\gamma-l]}\,\frac{dF}{d(x)}, \qquad (10)$$

and

$$F([\alpha][\beta][\gamma],\,\lambda p^{l},\,x)-F([\alpha][\beta][\gamma-l]\,\lambda x)=-\frac{x^{[l]}}{[\gamma-l]}\,\frac{dF}{dx}, \qquad (11)$$

and we obtain

$$(p^{\gamma-l}-1)\,F([\alpha][\beta][\gamma-l]\,\lambda,\,x)=p^{\gamma-l}F([\alpha][\beta][\gamma]\,p^{l}\lambda x)$$
$$-F([\alpha][\beta][\gamma]\,\lambda x)\,. \qquad (12)$$

From the differential equation we have

$$\left[\frac{dF}{d(x)}\right]_{x=1}\times\{p^{\gamma-\beta-l}[\beta+l]+p^{\gamma-\alpha-l}[\alpha]-[\gamma]\}$$
$$+p^{\gamma-\alpha-\beta-l}[\alpha][\beta]\,F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l})=0, \qquad (13)$$

also we have from (6), putting $\lambda=p^{\gamma-\alpha-\beta-l}$,

$$\left[\frac{dF}{d(x)}\right]_{x=1}=[\gamma-l]\{F([\alpha][\beta][\gamma-l]\,p^{\gamma-\alpha-\beta-l})-F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta})\}, \qquad (14)$$

and from (12) we have, putting $\lambda=p^{\gamma-\alpha-\beta-l}$,

$$(p^{\gamma-l}-1)\,F([\alpha][\beta][\gamma-l]\,p^{\gamma-\alpha-\beta-l})$$
$$=p^{\gamma-l}F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta})-F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l}). \qquad (15).$$

Eliminating $\left[\dfrac{dF}{d(x)}\right]_{x=1}$ between equations (13) and (14), then eliminating

$F([\alpha][\beta][\gamma]\, p^{\gamma-\alpha-\beta-l})$ by means of (15), we obtain

$$F([\alpha][\beta][\gamma]\, p^{\gamma-\alpha-\beta}) = \frac{[\gamma-l][\gamma-\alpha-\beta-l]}{[\gamma-\alpha-l][\gamma-\beta-l]} F([\alpha][\beta][\gamma-l]\, p^{\gamma-\alpha-\beta-l}), \tag{15A}$$

a relation between successive functions from which by repetition we have

$$F([\alpha][\beta][\gamma]\, p^{\gamma-\alpha-\beta}) = \underset{\kappa=\infty}{L} \frac{[\gamma-\alpha][\gamma-\alpha+l]\,..\,[\gamma-\alpha+(k-1)\,l].[\gamma-\beta][\gamma-\beta+l]\,..\,[\gamma-\beta+(k-1)\,l]}{[\gamma][\gamma+l]\,..\,[\gamma+(k-1)\,l]\,.\,[\gamma-\alpha-\beta]\,..\,[\gamma-\alpha-\beta+(k-1)\,l]} \times F([\alpha][\beta][\gamma+kl]\, p^{\gamma-\alpha-\beta+\kappa l}). \tag{16}$$

Each of the four infinite products on the right side of the above is convergent if $p^l < 1$ and $F([\alpha][\beta][\gamma+kl]\, p^{\gamma-\alpha-\beta+\kappa l}) = 1$ when k is infinite and $p^l < 1$.

So we write

$$\frac{\Gamma([\gamma])\,\Gamma([\gamma-\alpha-\beta])}{\Gamma([\gamma-\alpha])\,\Gamma([\gamma-\beta])} = F([\alpha][\beta][\gamma]\, p^{\gamma-\alpha-\beta})\,; \tag{17}$$

and by the help of (9) we deduce

$$F([\alpha][\beta][\gamma]\, p^{\gamma-\alpha-\beta-l} = \frac{\Gamma([\gamma])\,\Gamma([\gamma-\alpha-\beta])}{\Gamma([\gamma-\alpha])\,\Gamma([\gamma-\beta])} \cdot \frac{\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}}{[\gamma-\alpha-\beta-l]}. \tag{18}$$

If we consider p^l not as a simple element but as compounded of the form

$$p_1^{l_1} p_2^{l_2} \cdot\cdot\cdot\cdot\, p_s^{l_s},$$

then p^{a+rl} is

$$p_1^{a_1+r\,1},\ p_2^{a_2+rl_2}\ \cdot\cdot\cdot\cdot\ p_s^{a_2+rl_s}.$$

The series F can be transformed into various series such as

$$1 + p^{\gamma-\alpha-\beta} \frac{p^\alpha-1\,.\,p^\beta-z}{p^l-1\,.\,p^\gamma+z} + p^{2(\gamma-\alpha-\beta)} \frac{p^\alpha-1\,.\,p^{\alpha+l}-1\,.\,p^\beta+z\,.\,p^{\beta+l}+z}{p^l-1\,.\,p^{2l}-1\,.\,p^\gamma+z\,.\,p^{\gamma+l}+z} + \cdot\cdot\cdot\cdot, \tag{19}$$

$$1 + p^l \frac{p^\alpha-1\,.\,p^\beta-1}{p^l-1\,.\,p^\gamma+z} z + p^{2l} \frac{p^\alpha-1\,.\,p^{\alpha+l}-1\,.\,p^\beta-1\,.\,p^{\beta+l}-1}{p^l-1\,.\,p^{2l}-1\,.\,p^\gamma+z\,.\,p^{\gamma+l}+z} z^2 + \cdot\cdot\cdot\cdot \tag{20}$$

The following theorems obviously follow from the foregoing:

$$F([\alpha][\beta][\gamma]\, p^{\gamma-\alpha-\beta})\,.\,F([-\alpha][\beta][\gamma-\alpha]\, p^{\gamma-\beta}) = 1, \tag{21}$$

$$F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta}) \,.\, F([\alpha][-\beta][\gamma-\beta]\,p^{\gamma-\alpha}) = 1, \tag{22}$$

$$\begin{aligned} &F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l})\,F([-\alpha][\beta][\gamma-\alpha]\,p^{\gamma-\beta-l}) \\ &\qquad = \frac{\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}\{[\gamma-l]+p^{\gamma-\alpha-l}[-\beta]\}}{[\gamma-\alpha-\beta-l][\gamma-\beta-l]}, \end{aligned} \tag{23}$$

$$\begin{aligned} &F([\alpha][\beta][\gamma]\,p^{\gamma-\alpha-\beta-l})\,F([\alpha][-\beta][\gamma-\beta]\,p^{\gamma-\alpha-l}) \\ &\qquad = \frac{\{[\gamma-\beta-l]+p^{\gamma-l}[-\alpha]\}\{[\gamma-l]+p^{\gamma-\beta-l}[-\alpha]\}}{[\gamma-\alpha-\beta-l][\gamma-\alpha-l]}, \end{aligned} \tag{24}$$

$$\begin{aligned} &F([-\alpha][\beta][\gamma-\alpha]\,p^{\gamma-\beta-l}) \\ &\quad = \frac{[\gamma-\alpha-l]}{[\gamma-\beta-l]} \cdot \frac{\{[\gamma-\alpha-l]+p^{\gamma-l}[-\beta]\}\{[\gamma-l]+p^{\gamma-\alpha-l}[-\beta]\}}{\{[\gamma-\beta-l]+p^{\gamma-l}[-\alpha]\}\{[\gamma-l]+p^{\gamma-\beta-l}[-\alpha]\}} \\ &\qquad\qquad \times F([\alpha][-\beta][\gamma-\beta]\,p^{\gamma-\alpha-l}), \end{aligned} \tag{25}$$

$$\begin{aligned} &p^{\alpha}[\beta]\{F([\alpha][\beta][\gamma]\,\lambda x) - F([\alpha][\beta+l][\gamma]\,\lambda x\} \\ &\qquad = p^{\beta}[\alpha]\{F([\alpha][\beta][\gamma]\,\lambda x) - F([\alpha+l][\beta][\gamma]\,\lambda x)\} \end{aligned} \tag{26}$$

$$\qquad = \lambda^{p^{\alpha+\beta}} \frac{[\alpha][\beta]}{[\gamma]}\, x^{[l]}\, F([\alpha+l][\beta+l][\gamma+l]\,\lambda x^{p^l}) \tag{27}$$

$$\qquad = x^{[l]} p^{\alpha+\beta} \frac{dF}{d(x)} \,. \tag{28}$$

Other solutions of the differential equation give rise to the series

$$x^{[l-\gamma]}\, F([\alpha+l-\gamma][\beta+l-\gamma][2l-\gamma]\, x^{p^{l-\gamma}}\lambda),$$

and

$$\begin{aligned} &x^{[-\alpha]} + \frac{[\alpha][\gamma-\alpha-l]}{[l][\beta-\alpha-l]} \frac{p^{l-\alpha}}{\lambda} x^{[-l-\alpha]} \\ &\qquad + \frac{[\alpha][\alpha+l][\gamma-\alpha-l][\gamma-\alpha-2l]}{[l][2l][\beta-\alpha-l][\beta-\alpha-2l]} \cdot \frac{p^{2l-2\alpha}}{\lambda^2} x^{[-2l-\alpha]} + \ldots\,; \end{aligned}$$

the connection between the various solutions must be reserved for another paper.